

# COVARIANCE OF TIME-ORDERED PRODUCTS IMPLIES LOCAL COMMUTATIVITY OF FIELDS

O.W. Greenberg<sup>1</sup>

*Center for Theoretical Physics*

*Department of Physics*

*University of Maryland*

*College Park, MD 20742-4111*

## Abstract

We formulate Lorentz covariance of a quantum field theory in terms of covariance of time-ordered products (or other Green's functions). This formulation of Lorentz covariance implies spacelike local commutativity or anticommutativity of fields, sometimes called microscopic causality or microcausality. With this formulation microcausality does not have to be taken as a separate assumption.

## 1 Introduction

In formulating the basic assumptions of relativistic quantum field theory in terms of vacuum matrix elements of products of fields, A.S. Wightman [1, 2] chose to use unordered products of fields and to express the basic assumptions in terms of the Wightman distributions or, colloquially, functions. We will use commutativity (commutator) both for commutativity (commutator) for Bose fields and anticommutativity (anticommutator) for Fermi fields throughout this paper. In Wightman's formulation spacelike commutativity of fields is independent of relativistic covariance of the Wightman functions. It is generally accepted that local commutativity implies that vacuum matrix elements of time-ordered products of fields (or other Green's functions) are Lorentz covariant. The converse, that Lorentz covariance of vacuum matrix elements of time-ordered products of fields implies local commutativity of fields, does not seem to appear in the field theory literature. If we take Lorentz

---

<sup>1</sup>email address, owgreen@physics.umd.edu.

covariance of time-ordered products as the condition of Lorentz covariance of the field theory, then the demonstration of this converse, which we give below, means that local commutativity is not an independent assumption of the theory. This implies further that the spin-statistics and  $\mathcal{CPT}$  theorems also hold without further assumptions. All of this means that relativistic quantum field theory becomes a more coherent structure.

## 2 Proof that covariance of time-ordered products implies local commutativity of fields

Consider an arbitrary time-ordered function

$$\begin{aligned} \tau(x_{-N_L}, \dots, x_{-1}, x_{-0}, x_0, x_1, \dots, x_{N_R}) = \\ \langle 0 | T(\phi(x_{-N_L}) \cdots \phi(x_{-1}) \phi(x_{-0}) \phi(x_0) \phi(x_1) \cdots \phi(x_{N_R})) | 0 \rangle. \end{aligned} \quad (1)$$

There are  $N_L + N_R + 2$  points. Choose these points so that they correspond to a Jost point [3], i.e., the successive difference variables

$$\begin{aligned} \xi_{-N_L} = x_{-N_L} - x_{-N_L+1}, \dots, \xi_{-1} = x_{-1} - x_{-0}, \\ \xi = x_{-0} - x_0, \xi_1 = x_0 - x_1, \dots, \xi_{N_R} = x_{N_R-1} - x_{N_R} \end{aligned} \quad (2)$$

form a convex set that is totally spacelike.<sup>2</sup> That requires

$$\sum_{l=-1}^{-N_L} \lambda_l \xi_l + \lambda \xi + \sum_{r=1}^{N_R} \lambda_r \xi_r \sim 0, \forall \lambda_l \geq 0, \forall \lambda_r \geq 0, \lambda \geq 0, \quad (3)$$

and

$$\sum_{l=1}^{N_L-1} \lambda_l + \lambda + \sum_{r=1}^{N_R-1} \lambda_r > 0. \quad (4)$$

---

<sup>2</sup>My metric is  $\text{diag}(1, -1, -1, -1)$  and I use  $x^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$  as short for  $x \cdot x$ . I abbreviate  $x^2 < 0$ , i.e. spacelike, by  $x \sim 0$ ;  $x^2 > 0, x^0 > 0$ , i.e., the open positive light cone, by  $x \in V_+$ ; the open negative light cone by  $x \in V_-$ ; and the closed light cones by  $\bar{V}_\pm$ .

Because we chose the points to correspond to a Jost point, we can equate the time-ordered function of Eq.(1) to a Wightman function, i.e. to the vacuum matrix element of a product of fields,

$$\begin{aligned} \mathcal{W}(x_{-N_L}, \dots, x_{-1}, x_{-0}, x_0, x_1, \dots, x_{N_R}) \\ = \langle 0 | \phi(x_{-N_L}) \cdots \phi(x_{-1}) \phi(x_{-0}) \phi(x_0) \phi(x_1) \cdots \phi(x_{N_R}) | 0 \rangle. \end{aligned} \quad (5)$$

Choose the points  $x_i$  such that under a Lorentz transformation the time order of  $x_{-0}$  and  $x_0$  will reverse without changing the relative time order of any of the other points. Observer Lorentz covariance of the time-ordered product requires that these two different time orders must agree; this means the commutator must vanish at spacelike separation at Jost points in each Wightman function. These points will be sufficiently separated so that the same condition will hold for an open neighborhood of these points. Now add imaginary parts,  $\eta_i \in V_-$ , i.e. in or on the backward light cone in momentum space, to all the difference variables except  $\xi$  to define complex difference variables

$$\begin{aligned} \zeta_{-N_L} = \xi_{-N_L} + i\eta_{-N_L}, \dots, \zeta_{-1} = \xi_{-1} + i\eta_{-1}; \\ \zeta_1 = \xi_1 + i\eta_1, \dots, \zeta_{N_R} = \xi_{N_R} + i\eta_{N_R}. \end{aligned} \quad (6)$$

Note that we do not choose a complex variable to correspond to  $\xi = x_{-0} - x_0$ . The complex parts of the difference variables are associated with the vectors  $x_{-l}$  and  $x_r$  and not with the vectors  $x_{-0}$  and  $x_0$ . Then this condition, that

$$\langle 0 | \phi(x_{-N_L}) \cdots \phi(x_{-1}) [\phi(x_{-0}), \phi(x_0)]_{\mp} \phi(x_1) \cdots \phi(x_{N_R}) | 0 \rangle = 0, \quad (7)$$

can be analytically continued to all  $\zeta_{-l}$  and  $\zeta_r$ . The boundary values for  $\text{Im}\zeta_{-l} \rightarrow 0$  and  $\text{Im}\zeta_r \rightarrow 0$  are then equal for all  $x_{-l}$  and  $x_r$  and for  $x_{-0}$  spacelike with respect to  $x_0$ ; i.e. an arbitrary matrix element of the commutator  $[\phi(x_{-0}), \phi(x_0)]_{\mp} = 0$  for spacelike separation. Wightman's reconstruction theorem [4, 5] states that the set of all Wightman functions determines the quantum field theory up to unitary equivalence. We do not repeat the conditions for the reconstruction theorem here; they are the usual conditions of a relativistic quantum field theory and are described in detail in [1, 2, 5]. Thus the field is local and the theory obeys microcausality. Similar arguments show that the same conclusion follows from covariance of other

Greens functions.<sup>3</sup>

Choose the  $N_L + N_R + 2$  points as follows,

$$\begin{aligned} x_l &= (-la, 0, 0, -3la), 1 \leq l \leq N_L \\ x_r &= (ra, 0, 0, 3ra), 1 \leq r \leq N_R, x_{-0} = (0, 0, 0, -a), x_0 = (0, 0, 0, a). \end{aligned} \quad (9)$$

We can check that these points are Jost points and that there exists an open set containing them that contains only Jost points. Further, a small boost in the  $\pm z$ -direction will make  $x_0^0$  either greater than or less than  $x_{-0}^0$  without changing the time ordering of any of the other points. In order for the time-ordered function (distribution) to be independent of the frame of reference of the observer, the commutator must vanish at spacelike separation of  $x_{-0}$  and  $x_0$ .

Our analysis uses properties of the time-ordered product and the commutator at spacelike separation only and therefore does not depend on the difference between the time-ordered product and the  $T^*$ -product which occurs only at coincident points [6, 7, 8].

We emphasize that our demonstration does not assume pointlike form of the Lagrangian or Hamiltonian, therefore our argument holds for nonlocal theories in

---

<sup>3</sup>Note that if we had tried to make the same argument using an arbitrary time-ordered function we would have been able to conclude that

$$\begin{aligned} \langle 0 | T(\phi(x_{-N_L}) \cdots \phi(x_{-1}) \phi(x_{-0}) \phi(x_0) \phi(x_1) \cdots \phi(x_{N_R})) | 0 \rangle = \\ \langle 0 | T(\phi(x_{-N_L}) \cdots \phi(x_{-1}) \phi(x_0) \phi(x_{-0}) \phi(x_1) \cdots \phi(x_{N_R})) | 0 \rangle; \end{aligned} \quad (8)$$

however because the reconstruction theorem holds for Wightman functions, but not for time-ordered functions, we would not be able to conclude that we could reconstruct the quantum field theory. The merit of the argument using the time-ordered product at Jost points is that it allows the transition to Wightman functions. The analyticity of the Wightman functions in the neighborhood of a Jost function allows the conclusion that arbitrary matrix elements of the commutator (or anticommutator) vanish at spacelike separation and, using the reconstruction theorem, that the arbitrary matrix elements define a local quantum field theory. It may be that there is an analog of the reconstruction theorem for time-ordered products, but because the step functions that occur in the time-ordered product are discontinuous, the definition of a time-ordered product in terms of a Wightman function has ambiguities, and one can expect that such an analog will have technical complications.

which the fields in the Lagrangian or Hamiltonian enter at separated points in spacetime.

These arguments do not hold for parastatistics or quon fields because the time-ordered products for those fields are not simply related when neighboring fields are interchanged at spacelike separation. Observables in theories with parastatistics fields, but not in theories with quon fields, commute at spacelike separation, but the parastatistics fields themselves do not commute at spacelike separation.

The discussion given above for bose and fermi fields can be carried over word for word to conclude that observer covariance of time-ordered products of observable fields implies local commutativity (for observable fields always commutativity and not anticommutativity) of observable fields. This argument applies to observable fields in theories with bose, fermi, parabose and parafermi fields.

Acknowledgements: I am happy to thank Lev Okun, Peter Orland and Ching-Hung Woo for valuable discussions. This work was supported in part by the National Science Foundation, Award No. PHY-0140301.

## References

- [1] A.S. Wightman, Phys. Rev. 101, 860 (1956).
- [2] R.F. Streater and A.S. Wightman, *PCT, Spin and Statistics, and All That*, (Benjamin, New York, 1964).
- [3] R. Jost, Helv. Phys. Acta 30, 409 (1957).
- [4] R.F. Streater and A.S. Wightman, *op. cit.* pp117-126.
- [5] R. Jost, *The General Theory of Quantized Fields*, pp61-64, (American Mathematical Society, Providence, 1965).
- [6] R. Jackiw in *Current Algebra and Anomalies*, pp93-102, ed. S.B. Treiman, R. Jackiw, B. Zumino and E. Witten, (Princeton University Press, Princeton, 1985).
- [7] D.J. Gross and R. Jackiw, Nucl. Phys. B14, 269 (1969).

- [8] R.F. Dashen and S.Y. Lee, Phys. Rev. 187, 2017 (1969).
- [9] R.F. Streater and A.S. Wightman, *op. cit.* pp97-102, especially p102.